

Genuine non-congruence subgroups of Drinfeld modular groups

BY

A. W. MASON

*Department of Mathematics, University of Glasgow
Glasgow G12 8QW, Scotland, U.K.
e-mail: awm@maths.gla.ac.uk*

AND

ANDREAS SCHWEIZER¹

*Department of Mathematics,
Korea Advanced Institute of Science and Technology (KAIST),
Daejeon 305-701, South Korea
e-mail: schweizer@kaist.ac.kr*

Abstract

Let A be the ring of elements in an algebraic function field K over a finite field \mathbb{F}_q which are integral outside a fixed place ∞ . In an earlier paper we have shown that the *Drinfeld modular group* $G = GL_2(A)$ has automorphisms which map congruence subgroups to non-congruence subgroups. Here we prove the existence of (uncountably many) normal *genuine* non-congruence subgroups, defined to be those which remain non-congruence under the action of *every* automorphism of G . In addition, for all but finitely many cases we evaluate $\text{ngncs}(G)$, the smallest index of a normal genuine non-congruence subgroup of G , and compare it to the minimal index of an arbitrary normal non-congruence subgroup.

Key words: Drinfeld modular group; genuine non-congruence subgroup; non-standard automorphism

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Introduction

An important point in the theory of the classical modular group is the free product decomposition

$$PSL_2(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}.$$

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As a consequence, every finite group H that can be generated by an involution and an element of order 3 can be obtained as quotient group $H \cong SL_2(\mathbb{Z})/N$ for a normal finite index subgroup N of $SL_2(\mathbb{Z})$. Taking the compact Riemann surfaces that are the quotients of the extended complex upper half-plane by N resp. $SL_2(\mathbb{Z})$ one obtains a Galois cover of $\mathbb{P}^1(\mathbb{C})$ with prescribed Galois group H . Some of these quotients also furnish examples of Riemann surfaces whose automorphism group is big compared to their genus.

Now for congruence subgroups N , the composition factors of $SL_2(\mathbb{Z})/N$ can only be cyclic or $PSL_2(\mathbb{F}_p)$. So for most H the group N necessarily is a non-congruence subgroup.

For the analogue in positive characteristic, let K be an algebraic function field of one variable with constant field $k = \mathbb{F}_q$. As usual we assume that k is algebraically closed in K . Let ∞ be a fixed place of K of degree δ and let A be the ring of all those elements of K which are integral outside ∞ . The simplest example: If K is a rational function field over k and $\delta = 1$, then $A \cong k[t]$, a polynomial ring over k .

Our focus of attention here are the *Drinfeld modular groups* $G = GL_2(A)$. These groups play a central role [G2] in the theory of *Drinfeld modular curves*, analogous to that of the classical modular group $SL_2(\mathbb{Z})$ in the theory of modular curves and modular forms.

From the quotients of the *Drinfeld upper half-plane* by finite index subgroups N of G one obtains algebraic curves in positive characteristic. If N is normal in G and contains the center Z of G , then G/N is a subgroup of the automorphism group of that curve. Again, depending on what one would want G/N to be, in most cases N has to be a non-congruence subgroup of G . (See Lemma 2.5).

In an earlier paper [MS3] we have shown that $SL_2(A)$ has automorphisms which map (some) congruence subgroups to non-congruence subgroups. Such automorphisms can be readily extended to G (Theorem 2.2). They were originally introduced for the special case $A = k[t]$ by Reiner [R]. We extend a terminology used in [MS3].

Definition. We call an automorphism Ψ of an arithmetic group X **non-standard** if there exists a congruence subgroup C of X such that $\Psi(C)$ is non-congruence. Otherwise Ψ is called **standard**.

Non-standard automorphisms have been used in [MS3] to construct many non-congruence subgroups with prescribed properties. However, by construction, these non-congruence subgroups are subject to the same group-theoretic restrictions as congruence subgroups. In particular, if such a group N is normal, the possible composition factors of G/N are known (Lemma 2.5). So if we want, just as an example, the simple group A_7 to be among these composition factors, then N cannot be a congruence subgroup or obtained from one by a non-standard automorphism. What we need must be a “genuine” non-congruence subgroup in the following sense.

Definition. Let S be a non-congruence subgroup of an arithmetic group X . We call S **genuine** if and only if $\Psi(S)$ is a non-congruence subgroup, for all $\Psi \in \text{Aut}(X)$.

Actually, using composition factors of G/N that cannot occur for congruence subgroups, the existence of genuine non-congruence subgroups of Drinfeld modular group can be proved relatively easily from known results.

Theorem A. *With every G there corresponds a finite set of A -ideals $\mathcal{S} = \mathcal{S}(G)$ with the property that, for all $\mathfrak{q} \notin \mathcal{S}$, there exist infinitely many normal genuine non-congruence subgroups N for which*

$$\text{ql}(N) = \mathfrak{q},$$

where $\text{ql}(N)$ is the quasi-level of N .

The definition of the *quasi-level* of a subgroup of G extends the classical definition of the *level* of a subgroup (originally applied to the modular group $SL_2(\mathbb{Z})$ or $PSL_2(\mathbb{Z})$). The existence of uncountably many normal genuine non-congruence subgroups of any G follows in similar way from an earlier result of Lubotzky [L, Theorem A(ii)] which states that G always has a finite index subgroup mapping onto the free group of rank 2. Lubotzky uses this result to prove [L, Theorem B(ii)] that \hat{F}_ω , the free profinite group on countably many generators, is a closed subgroup of $C(SL_2(A))$, the *congruence kernel* of $SL_2(A)$. The relationship between \hat{F}_ω and $C(SL_2(A))$ is further explored in [M3] and [MPSZ].

In previous papers [M4], [MS1], [MS2] we determined the smallest index of a non-congruence subgroup of $SL_2(A)$. Here we turn our attention to $\text{ngnics}(G)$, the smallest index of a normal *genuine* non-congruence subgroup of G . Our second main result (in Section 5) is the following.

Theorem B. *In all but finitely many cases*

$$\text{ngnics}(G) = \text{m}(G) = 2,$$

where $\text{m}(G)$ is the smallest index of a proper subgroup of G .

Note that this implies that the minimal index of a (not necessarily normal) genuine non-congruence subgroup is then also 2. We also determine $\text{ngnics}(G)$ for a number of cases for which Theorem B does not hold. In particular, for the very important special case where A is a polynomial ring $\mathbb{F}_q[t]$ we are able to show that $\text{ngnics}(G)$ is strictly bigger than the minimal index of a normal non-congruence subgroup of G . See Section 6 for more and more precise results.

Although non-standard automorphisms exist for every Drinfeld modular group this is not true for other important arithmetic groups like $SL_2(\mathbb{Z})$ and the Bianchi groups $SL_2(\mathcal{O})$, where \mathcal{O} is the ring of integers in an imaginary quadratic number field. For these groups their automorphisms are known [HR], [SV] to be “standard”. So for them the definition of a genuine non-congruence subgroup is redundant, i.e. all their non-congruence subgroups are genuine.

Other known results in characteristic zero include the following. It is known [D, Proposition 4.3] that $SL_2(\mathbb{Z})$ is a characteristic subgroup of $GL_2(\mathbb{Z})$ which, in view of [HR], implies that every non-congruence subgroup of $GL_2(\mathbb{Z})$ is genuine. It is worth noting that results for congruence subgroups of groups of type SL_2 and GL_2 do not necessarily apply to the corresponding projective groups PSL_2 and PGL_2 . For example it is known that every non-congruence subgroup of $PSL_2(\mathbb{Z})$ is genuine by [D, Corollary 4.4 (2)]. On the other hand, it is also known [JT] that $PGL_2(\mathbb{Z})$ has a non-standard automorphism which therefore cannot fix $PSL_2(\mathbb{Z})$. Hence $PSL_2(\mathbb{Z})$ is not characteristic in $PGL_2(\mathbb{Z})$.

Notation

$k = \mathbb{F}_q$	the finite field of order $q = p^n$;
K	an algebraic function field of one variable with constant field \mathbb{F}_q ;
$g = g(K)$	the genus of K ;
∞	a chosen place of K ;
δ	the degree of the place ∞ ;
ν	the discrete valuation of K defined by ∞ ;
\mathcal{T}	the Bruhat-Tits tree of $GL_2(K_\infty)$;
A	the ring of all elements of K that are integral outside ∞ ;
G	the group $GL_2(A)$;
Γ	the group $SL_2(A)$;
Z	the centre of G ;
X	the groups G, Γ .

1. Subgroups defined by ideals

Let \mathfrak{q} be an A -ideal. It is known that A/\mathfrak{q} is *finite*, when \mathfrak{q} is non-zero. We recall that by the well-known product formula $\nu(a) \leq 0$, for all $a \in A$, and that $\nu(a) = 0$ if and only if $a \in k^*$.

For each $\alpha, \beta \in k^*, a \in A$, we put

$$L(\alpha, \beta, a) := \begin{bmatrix} \alpha & a \\ 0 & \beta \end{bmatrix}$$

and

$$T(a) := L(1, 1, a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}.$$

For every subset S of A we put

$$T(S) := \{T(s) : s \in S\}.$$

For each subgroup H of G and $g \in G$ we denote the conjugate gHg^{-1} by H^g .

Definition. We define the **quasi-level** of H to be

$$\text{ql}(H) := \{h \in A : T(h) \in H^g, \text{ for all } g \in G\}.$$

The **level** of H is the biggest A -ideal contained in $\text{ql}(H)$.

Clearly $\text{ql}(H)$ is an additive subgroup of A and, by considering conjugates by the elements $\text{diag}(\alpha, 1)$, where $\alpha \in k^*$, it is clear that it is also a vector space over k .

In [MS3] the “quasi-level” of $H \leq \Gamma$ is defined to be

$$\text{ql}(H)^* := \{h \in A : T(h) \in H^g, \text{ for all } g \in \Gamma\}.$$

It is easily shown that the definitions are equivalent.

Lemma 1.1. $\text{ql}(H)^* = \text{ql}(H)$.

Proof. Clearly $\text{ql}(H) \leq \text{ql}(H)^*$. By conjugating with the matrices $\text{diag}(\alpha, \alpha^{-1})$ and using the fact that every element in a finite field is a sum of two squares, it follows that $\text{ql}(H)^*$ is a vector space over k .

Now let $h \in \text{ql}(H)^*$ and $g \in G$. Then $g = dg'$, where $d = \text{diag}(\beta, 1)$ (with $\beta \in k^*$) and $g' \in \Gamma$. Then, from the above, $T(h\beta^{-1}) \in H^{g'}$ and so $T(h) = dT(h\beta^{-1})d^{-1} \in H^g$. \square

We define the Borel subgroup of G

$$B_2(A) = G_\infty = \{L(\alpha, \beta, a) : \alpha, \beta \in k^*, a \in A\}.$$

Definition. For every ideal \mathfrak{q} of A we define

$$\Delta(\mathfrak{q}) := \text{the normal subgroup of } \Gamma \text{ generated by } T(\mathfrak{q}).$$

A subgroup H of G is said to have *non-zero level* if $\Delta(\mathfrak{q}) \subseteq H$, for some $\mathfrak{q} \neq \{0\}$. Otherwise H is said to have *level zero*.

As in the proof of Lemma 1.1 one shows that $\Delta(\mathfrak{q})$ is also normal in G . So $\Delta(\mathfrak{q})$ can also be defined as the normal subgroup of G generated by $T(\mathfrak{q})$.

Definition. Let

$$\Gamma(\mathfrak{q}) = \{M \in \Gamma : M \equiv I_2 \pmod{\mathfrak{q}}\}.$$

A subgroup C of G is said to be a **congruence subgroup** if $\Gamma(\mathfrak{q}') \leq C$, for some $\mathfrak{q}' \neq \{0\}$. Such a subgroup is necessarily of finite index in G . A finite index subgroup of G which is not congruence is called a **non-congruence subgroup**.

We will make use of the following properties of these subgroups. The first [B, (9.3) Corollary, p.267] plays an important role in determining whether or not a finite index subgroup of G is congruence (the so-called *congruence subgroup problem*).

Lemma 1.2. *Let $\mathfrak{q}_1, \mathfrak{q}_2$ be A -ideals, where $\mathfrak{q}_2 \neq \{0\}$. Then*

$$\Delta(\mathfrak{q}_1) \cdot \Gamma(\mathfrak{q}_2) = \Gamma(\mathfrak{q}_1 + \mathfrak{q}_2).$$

From this one easily obtains the following classical criterion for being a congruence subgroup.

Corollary 1.3. *Let H be a subgroup of G with $\Delta(\mathfrak{q}) \subseteq H$ for some non-zero ideal \mathfrak{q} of A . Then H is a congruence subgroup if and only if H contains $\Gamma(\mathfrak{q})$.*

Proof. If H is a congruence subgroup, then by definition it contains $\Gamma(\mathfrak{q}')$ for some non-zero ideal \mathfrak{q}' . So by Lemma 1.2 it contains $\Gamma(\mathfrak{q} + \mathfrak{q}')$, which contains $\Gamma(\mathfrak{q})$. The converse is clear. \square

In view of this criterion, the existence of non-congruence subgroups of non-zero level is a consequence of the following result.

Lemma 1.4. *There exists an epimorphism*

$$\Gamma(\mathfrak{q})/\Delta(\mathfrak{q}) \twoheadrightarrow F_r,$$

where F_r is the free group of (finite) rank $r = r(\mathfrak{q}) = \text{rk}_{\mathbb{Z}}(\Gamma(\mathfrak{q}))$, the torsion-free rank of the abelianization of $\Gamma(\mathfrak{q})$. Moreover

$$r(\mathfrak{q}) \rightarrow \infty \text{ as } A/\mathfrak{q} \rightarrow \infty.$$

Proof. Let $\Lambda(\mathfrak{q})$ be the subgroup of G generated by $\Gamma(\mathfrak{q}) \cap G_v$, for all $v \in \text{vert}(\mathcal{T})$. Then $\Lambda(\mathfrak{q}) \trianglelefteq G$ and, from the theory of groups acting on trees [Se, Corollary 1, p.55], it is known that

$$\Gamma(\mathfrak{q})/\Lambda(\mathfrak{q}) \cong F_{r(\mathfrak{q})},$$

the fundamental group of the quotient graph $\Gamma(\mathfrak{q}) \backslash \mathcal{T}$, which is known [Se, Corollary 4, p.108] to have *finite* rank $r(\mathfrak{q})$. Again from the presentation [Se, p.42] of $\Gamma(\mathfrak{q})$ derived from its action on \mathcal{T} , together with [Se, Proposition 2, p.76], it follows that $\Lambda(\mathfrak{q})$ is the subgroup of $\Gamma(\mathfrak{q})$ generated by its torsion elements. Hence $\Delta(\mathfrak{q}) \leq \Lambda(\mathfrak{q})$. Given \mathfrak{q}_1 and any \mathfrak{q}_2 for which $\mathfrak{q}_2 \leq \mathfrak{q}_1$, it is clear that $F_{r(\mathfrak{q}_2)} \leq F_{r(\mathfrak{q}_1)} \cap \Gamma(\mathfrak{q}_2)$. By means of the Schreier formula we can always choose \mathfrak{q}_2 such that $r(\mathfrak{q}_2) > r(\mathfrak{q}_1)$. \square

2. Non-standard automorphisms

For each non-negative integer n , we put

$$A(n) := \{x \in A : \nu(x) \geq -n\}.$$

Then $A(n)$ is a *finite-dimensional* vector space over k , whose dimension is determined by the *Riemann-Roch Theorem*. (See [St, Theorem 1.5.15, p.30].) Obviously $k \subseteq A(n)$. In particular $A(0) = k$, by [St, Corollary 1.1.20, p.8].

We put

$$G_n := \{L(\alpha, \beta, a) : \alpha, \beta \in k^*, a \in A(n)\}.$$

Then G_n is a finite subgroup of G . We note that

$$G_\infty = \bigcup_{n \geq 0} G_n.$$

Serre's decomposition theorem [Se, Theorem 10, p.119] shows that G_∞ is a (non-trivial) factor in a decomposition of G as an amalgamated product of a pair of its subgroups. See [MS3, Theorems 2.1, 2.2].

Theorem 2.1. *Let n be the smallest non-negative integer for which*

$$\delta n \geq 2g - 1.$$

Then there exists a subgroup H of G , such that

$$G = G_\infty *_L H,$$

where $L = G_{n_0}$. Moreover

$$\dim_k(A(n_0)) = n_0 \delta + 1 - g.$$

Note that $n_0 = 0$ when $g = 0$. Otherwise $n_0 > 0$.

Definition. Let $\phi : A \rightarrow A$ by any k -automorphism of the k -vector space A which fixes the elements of $A(n_0)$ (including k). Then ϕ induces an automorphism Φ of G_∞ defined by

$$\Phi : L(\alpha, \beta, a) \mapsto L(\alpha, \beta, \phi(a)),$$

where $\alpha, \beta \in k^*, a \in A$.

The following is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let ϕ be any k -automorphism of A which fixes the elements of $A(n_0)$. The map*

$$\Phi(g) = \begin{cases} L(\alpha, \beta, \phi(a)) & , \quad g = L(\alpha, \beta, a) \in G_\infty \\ g & , \quad g \in H \end{cases}$$

extends to an automorphism of G (and Γ).

As shown in [R] and [MS3] in most cases such an automorphism is *non-standard*. *Standard* automorphisms include inner automorphisms, the contragredient map $M \mapsto (M^T)^{-1}$, twists with certain determinant characters, i.e. $M \mapsto \chi(\det(M))M$ where $\chi : k^* \rightarrow k^*$ is a group homomorphism (with the property that $\chi(\alpha^2) = \alpha^{-1}$ if and only if $\alpha = 1$), or automorphisms derived from *ring*-automorphisms of A . Such a ring-automorphism ψ of A induces the automorphism Ψ of G , defined by

$$\Psi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} \psi(a) & \psi(b) \\ \psi(c) & \psi(d) \end{bmatrix}.$$

Clearly every such ring-automorphism maps A -ideals to A -ideals.

Also note that the standard automorphism of $GL_2(\mathbb{F}_q[t])$ induced by the ring automorphism $t \mapsto t+1$ of $\mathbb{F}_q[t]$ can also be obtained via the construction in Theorem 2.2 from a suitable \mathbb{F}_q -vector space automorphism ϕ . This shows that not every automorphism in Theorem 2.2 is necessarily non-standard.

For the case $A = \mathbb{F}_q[t]$ it is known [R] that the standard automorphisms listed above together with the automorphisms from Theorem 2.2 generate $\text{Aut}(G)$. This is the only case for which $\text{Aut}(G)$ is known.

We record two obvious properties.

Lemma 2.3. *For every subgroup S of G and any automorphism Φ of G defined by ϕ (as in Theorem 2.2) we have*

$$\text{ql}(\Phi(S)) = \phi(\text{ql}(S)).$$

In particular

$$\text{ql}(\Phi(S)) = A \Leftrightarrow \text{ql}(S) = A.$$

Note however that an *arbitrary* automorphism of G or Γ need not map subgroups of quasi-level A to subgroups of quasi-level A .

Example 2.4. Let

$$A = \mathbb{F}_2[x, y] \quad \text{with} \quad y^2 + xy = x^3 + x^2 + x.$$

Then by the results of Takahashi [T] we have

$$GL_2(A) \cong \Delta(\infty) * \Delta(0) * \Delta(1)$$

in the notation of [MS2, Theorem 5.3] (not of Section 1 of the current paper). Explicitly, $\Delta(1) \cong \mathbb{Z}/3\mathbb{Z}$ and

$$\Delta(\infty) = GL_2(\mathbb{F}_2) \underset{B_2(\mathbb{F}_2)}{*} B_2(A)$$

and $\Delta(0)$ is isomorphic (as an abstract group) to $\Delta(\infty)$. So there is a automorphism σ of $GL_2(A)$ that interchanges $\Delta(0)$ and $\Delta(\infty)$.

From the free product we can construct normal finite index subgroups H containing $\Delta(0)$ and $\Delta(1)$ with practically any prescribed quasi-level. Then $\sigma(H)$ contains $\Delta(\infty)$, and hence has quasi-level A .

Before proceeding we require a further definition.

Definition. For each ideal \mathfrak{q} let

$$Z(\mathfrak{q}) = \{X \in G : X \equiv \alpha I_2 \pmod{\mathfrak{q}} \text{ for some } \alpha \in A\}.$$

It is clear that $Z(\mathfrak{q})/\Gamma(\mathfrak{q}^2)$ is abelian.

Lemma 2.5. *Let N be a normal congruence subgroup of index n in G and let Ψ be any automorphism of G . Then the (simple) factors in a composition series of $G/\Psi(N)$ are either cyclic of prime order or are isomorphic to some $PSL_2(\mathbb{F}_{q^s})$, where $s \geq 1$.*

Moreover, if the level of N is divisible by a prime \mathfrak{p} with $|A/\mathfrak{p}| > 3$, then at least one factor in this composition series is of the latter type.

Proof. It is known [M2, Theorem 3.14] that, for some non-zero ideal \mathfrak{q}_0

$$\Gamma(\mathfrak{q}_0^2 \mathfrak{a}^2) \leq N \leq Z(\mathfrak{q}_0),$$

where \mathfrak{a} is the product of all prime ideals of A of index 2 or 3. If none exists we put $\mathfrak{a} = A$. We may confine our attention therefore to the composition factors of the groups $Z(\mathfrak{q})/\Gamma(\mathfrak{q}^2 \mathfrak{a}^2)$ and $Z(\mathfrak{q})/Z(\mathfrak{q}\mathfrak{p})$, where \mathfrak{p} as usual is prime.

We write $\mathfrak{a} = \mathfrak{a}_1 \mathfrak{a}_2$, where $\mathfrak{a}_1 + \mathfrak{q} = A$ and \mathfrak{a}_2 divides \mathfrak{q} . From standard results (like Lemma 1.2) it follows that

$$\Gamma(\mathfrak{q})/\Gamma(\mathfrak{q}^2 \mathfrak{a}^2) \cong (\Gamma(\mathfrak{q})/\Gamma(\mathfrak{q}^2 \mathfrak{a}_2^2)) \times \prod_{\mathfrak{p}|\mathfrak{a}_1} \Gamma/\Gamma(\mathfrak{p}^2).$$

Now the first group in the decomposition is metabelian from above and each $\Gamma/\Gamma(\mathfrak{p}^2)$ is a soluble group of order $m^4(m^2 - 1)$, where $m = 2, 3$.

We now consider the group $\Gamma(\mathfrak{q})/\Gamma(\mathfrak{q}\mathfrak{p})$. If \mathfrak{p} divides \mathfrak{q} it is abelian. On the other hand if $\mathfrak{p} + \mathfrak{q} = A$ then

$$\Gamma(\mathfrak{q})/\Gamma(\mathfrak{q}\mathfrak{p}) \cong \Gamma/\Gamma(\mathfrak{p}) \cong SL_2(\mathbb{F}_{q^s}),$$

where $\mathbb{F}_{q^s} = A/\mathfrak{p}$. The first part follows.

Under the condition of the second statement we have

$$N \leq Z(\mathfrak{p})$$

again by [M2, Theorem 3.14]. We note that

$$G/Z(\mathfrak{p}) \hookrightarrow PGL_2(\mathbb{F}_{q^s}),$$

where $\mathbb{F}_{q^s} = A/\mathfrak{p}$. It is well-known that $PSL_2(\mathbb{F}_{q^s})$ is contained in this embedding. When $q^s > 3$ the latter group is simple. The second part follows. \square

If $q > 3$, the condition in the second statement of Lemma 2.5 is of course automatic for all congruence subgroups except those containing Γ . We record a restricted version of this lemma.

Lemma 2.6. *Let N be a proper normal congruence subgroup of Γ and let Ψ be any automorphism of Γ . Then the factors in a composition series of $\Gamma/\Psi(N)$ are as in Lemma 2.5. Moreover if the level of N is divisible by a prime \mathfrak{p} with $|A/\mathfrak{p}| > 3$, then at least one factor in this composition series is of the latter type.*

3. Genuine non-congruence subgroups

Notation. For the remainder of this paper X will always denote G or Γ .

Definition. A finite index subgroup S of X is said to be a **genuine non-congruence subgroup** of X if $\Psi(S)$ is a non-congruence subgroup, for all $\Psi \in \text{Aut}(X)$.

The following straightforward result enables us in many instances to assume that a given genuine non-congruence subgroup is normal.

Lemma 3.1. *A finite index subgroup of X is a genuine non-congruence subgroup if and only if its core in X is a genuine non-congruence subgroup of X .*

Remark 3.2. Note however that we cannot be sure whether a genuine non-congruence subgroup H of Γ automatically is a genuine non-congruence subgroup of G , as theoretically G might have non-standard automorphisms (other than those discussed in Theorem 2.2) that do not respect Γ .

Similarly, if N is a (normal) genuine non-congruence subgroup of G , we cannot be sure whether $N \cap \Gamma$ is a genuine non-congruence subgroup of Γ , as theoretically there might be automorphisms of Γ that do not extend to automorphisms of G .

We now make use of Lemma 1.4 to prove the existence of genuine non-congruence

subgroups.

Theorem 3.3. *For all but finitely many \mathfrak{q} there exist infinitely many normal genuine non-congruence subgroups N of X for which*

$$\mathrm{ql}(N) = \mathfrak{q}.$$

Proof. By Lemma 1.4 there exists an epimorphism

$$\Gamma(\mathfrak{q})/\Delta(\mathfrak{q}) \twoheadrightarrow F_2,$$

for all but finitely many \mathfrak{q} . We note that if M is a finite index subgroup of X and

$$\Delta(\mathfrak{q}) \leq M \not\leq \Gamma(\mathfrak{q}),$$

then M is non-congruence by Corollary 1.3.

Let H be any finite group that can be generated by 2 elements and that has a non-cyclic simple composition factor that is not isomorphic to any $PSL_2(\mathbb{F}_{q^s})$. For example, we can always take $H = S_n$ with $n \geq 7$. Or if q is different from 2, 4, 5, we can even take $H = A_5 \cong PSL_2(\mathbb{F}_5) \cong PSL_2(\mathbb{F}_4)$. Then there exists $M \leq \Gamma(\mathfrak{q})$, where $M \geq \Delta(\mathfrak{q})$, for which

$$\Gamma(\mathfrak{q})/M \cong H.$$

Let N be the core of M in X . Then $\Delta(\mathfrak{q}) \leq N \leq \Gamma(\mathfrak{q})$ so that $\mathrm{ql}(N) = \mathfrak{q}$. In addition N is genuine by Lemmas 2.5, 2.6. \square

The subgroups in Theorem 3.3 have all non-zero level. By an earlier method we can prove the following.

Theorem 3.4. *There exist uncountably many normal genuine non-congruence subgroups of X of level zero.*

Proof. As in the proof of Theorem 3.3 we choose \mathfrak{q} and $N \trianglelefteq X$ so that $\Delta(\mathfrak{q}) \leq N \leq \Gamma(\mathfrak{q})$ and A_7 , say, is a factor in the composition series of X/N . Now choose an ideal \mathfrak{q}_0 so that $A = V_0 \oplus \mathfrak{q}_0$, where V_0 is a finite-dimensional space containing $A(n_0)$ (from Theorem 2.1).

Now, if $\mathfrak{q}_1 \leq \mathfrak{q}_2$ and $\mathfrak{q}_1 \neq \{0\}$, then the natural map

$$\Gamma(\mathfrak{q}_1)/\Delta(\mathfrak{q}_1) \longrightarrow \Gamma(\mathfrak{q}_2)/\Delta(\mathfrak{q}_2)$$

is *surjective* by Lemma 1.2. So replacing \mathfrak{q} with $\mathfrak{q} \cap \mathfrak{q}_0$ we may assume that $\mathfrak{q} = \mathfrak{q}_0$. Let W be one of the uncountably many subspaces of A not containing any non-zero A -ideal for which $A = V_0 \oplus W$. Then we can find a non-standard isomorphism Φ of X for which $\mathrm{ql}(\Phi(N)) = W$. \square

However as we now show many genuine non-congruence subgroups of X of quasi-level A do exist. Let

$$X_V = \langle X_v : v \in \text{vert}(\mathcal{T}) \rangle.$$

By [Se, Proposition 2, p.76] X_V is the subgroup of X generated by all its torsion elements and so is invariant under every automorphism of X . It follows that $\Delta(A) \leq X_V$. In addition

$$X/X_V \cong F_{r(X)},$$

where $F_{r(X)}$ is the fundamental group of the quotient graph $X \setminus \mathcal{T}$. See [Se, Corollary 1, p.55]. The rank $r(X)$ is known to be finite [Se, Corollary 4, p.108]. In particular $r(X) = 0$ if and only if $X \setminus \mathcal{T}$ is a *tree*. Moreover there are formulae for $r(X)$ involving δ, q and values of the *L-polynomial* of K ([G1], [G2, p.73], or see [MS1, p.33].) From these it is clear that, for any fixed g , $r(X) \rightarrow \infty$, as $\delta, q \rightarrow \infty$.

It is clear that $r(\Gamma) \geq r(G)$. The rank zero cases are known precisely [MS1, Theorem 2.10]. For convenience we record them.

Theorem 3.5.

- (i) $r(G) = 0$ if and only if $(g, \delta) = (1, 1), (0, 1), (0, 2)$ or $(0, 3)$.
- (ii) $r(\Gamma) = 0$ if and only if $(g, \delta) = (0, 1), (0, 2)$ or (when q is even) $(0, 3), (1, 1)$.

Lemma 3.6. *Let S be any proper finite index subgroup of X containing X_V . Then S is a genuine non-congruence subgroup of X with $\text{ql}(S) = A$.*

Proof. Recall that $\Psi(X_V) = X_V$ for all $\Psi \in \text{Aut}(X)$. Any congruence subgroup containing X_V must contain $\Gamma X_V = X$ by Lemma 1.2. \square

Corollary 3.7. *Suppose that $r = r(X) > 0$ and that H is any finite group with at most r generators. Then there exists a normal genuine non-congruence subgroup N of X of level A with*

$$X/N \cong H.$$

When $r(X) = 0$ there need not be any non-congruence subgroups of level A . Consider, for example, the case where $(g, \delta) = (0, 1)$. Then $A = k[t]$, a euclidean ring. In this case then $\Delta(A) = \Gamma$ and $X_V = X$.

On the one hand, Lemmas 2.5, 2.6 provide necessary conditions for a non-congruence subgroup to be not genuine. On the other hand, Corollary 3.7 enables us to show that these conditions are not sufficient.

Example 3.8. We provide a simple illustration of Corollary 3.7. Consider one of the many G with $r(G) \geq 4$. Let \mathfrak{p} be any prime A -ideal. Then, by Lemma 1.2,

$$G/\Gamma(\mathfrak{p}) \cong SL_2(A/\mathfrak{p}) \rtimes k^*.$$

Now A/\mathfrak{p} is a field $\mathbb{F}_{q'}$, for some power q' of q , and so $SL_2(A/\mathfrak{p})$ is generated by all $T(a)$ and $T(a)^T$. Let λ be a generator of $\mathbb{F}_{q'}^*$. Then $G/\Gamma(\mathfrak{p})$ is generated by $T(1)$, $\text{diag}(\lambda, \lambda^{-1})$, $\text{diag}(\mu, 1)$, where μ generates k^* and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. (Recall that every element in a finite field is a sum of 2 squares.) Then there exists $N \leq G$, with $G_V \leq N$, such that

$$G/N \cong G/\Gamma(\mathfrak{p}).$$

In some cases, of course, the rank restriction here can be weakened. For example, if $q' = q$, the group $G/\Gamma(\mathfrak{p})$ is 3-generated. And if $A/\mathfrak{p} \cong \mathbb{F}_p$, a prime field, then $\Gamma/\Gamma(\mathfrak{p}) \cong SL_2(\mathbb{F}_p)$ is even 2-generated (as a quotient of $SL_2(\mathbb{Z})$).

4. Some immediate criteria for being genuine

In this section we show that sometimes the index of a subgroup in itself can show that it is genuine non-congruence.

Proposition 4.1. *Let N be a proper normal subgroup of index n in G , where $\gcd(n, q) = 1$. Suppose further that there exists $S \leq N$ such that*

$$S \cong k^* \times k^*.$$

Then N is a genuine non-congruence subgroup of G .

Proof. Clearly $\Delta(A) \leq \Psi(N)$ for any automorphism Ψ of G . By Maschke's theorem applied to (the abelian group) $\Psi(S)$, there exists $g \in GL_2(K)$ such that

$$g\Psi(S)g^{-1} = D,$$

where D is the set of all diagonal matrices in $GL_2(k)$. Hence $\Psi(N).\Gamma = G$ and so $\Psi(N)$ is a non-congruence subgroup of G by Corollary 1.3. \square

Remark 4.2. If q is even or if $n > 2$, the condition $S \leq N$ in Proposition 4.1 can be replaced by $Z \leq N$ as then $Z \leq \Psi(N)$ for any $\Psi \in \text{Aut}(G)$. However, it is not clear whether conditions like $\det(N) = k^*$ or $N.\Gamma = G$ would suffice, as their behaviour under Ψ is not obvious.

The version of Proposition 4.1 for Γ is simpler.

Proposition 4.3. *Let N be a proper normal subgroup of index n in Γ , where $\gcd(n, q) = 1$. Then N is a genuine non-congruence subgroup of Γ .*

The following two results are easy conclusions of Lemmas 2.5 and 2.6.

Proposition 4.4. *Suppose that $q > 3$ and that N is a normal subgroup of index n in G , where $n \nmid (q - 1)$. If $|PSL_2(\mathbb{F}_q)| \nmid n$, then N is a genuine non-congruence subgroup of G .*

Proposition 4.5. *Suppose that $q > 3$ and that N is a proper normal subgroup of index n in Γ . If $|PSL_2(\mathbb{F}_q)| \nmid n$, then N is a genuine non-congruence subgroup of Γ .*

Note that Propositions 4.4, 4.5 hold in particular if $\gcd(n, q) = 1$ or $\gcd(n, q \pm 1) = 1$. The restrictions on q are necessary. When $q = 2, 3$ and $A = k[t]$ it is well-known that X has normal congruence subgroups of index q . Moreover it is known [MS1, Lemma 3.1] that for these cases X has non-congruence subgroups of index q which are *not* genuine.

Notation. For the case where a group H has proper finite index subgroups, we denote by $m(H)$ (> 1) the smallest index of such a subgroup.

It is a classical result (originally due to Galois) that $m(SL_2(\mathbb{F}_q)) = q + 1$ for $q > 11$ and $q = 4, 8$. Otherwise this index is q unless $q = 9$ in which case it is 6.

For non-normal subgroups of Γ we can now prove the following.

Proposition 4.6. *Suppose that $q > 3$ and that H is a proper subgroup of Γ for which*

$$|\Gamma : H| < m(SL_2(\mathbb{F}_q)).$$

Then H is a genuine non-congruence subgroup of Γ .

Proof. Let $S = SL_2(\mathbb{F}_q)$. Then, for each $g \in \Gamma$,

$$|S : S \cap H^g| \leq |\Gamma : H^g|.$$

It follows that $S \leq H^g$, and hence that S is contained in the core of H . By [MS3, Lemma 3.2] this implies $\Delta(A) \leq H$. By Corollary 1.3, H then is non-congruence. We can repeat the argument with $\Psi(H)$, for all $\Psi \in \text{Aut}(\Gamma)$. \square

5. The minimum index of a genuine non-congruence subgroup

The first two of the following appear in [MS1] and [MS2].

Definitions.

- (i) $\text{ncs}(X) = \min\{|X : S| : S \leq X, S \text{ noncongruence}\}.$

- (ii) $\text{nncs}(X) = \min\{|X : S| : S \leq X, S \text{ normal, noncongruence}\}.$
- (iii) $\text{gncs}(X) = \min\{|X : S| : S \leq X, S \text{ genuine noncongruence}\}.$
- (iv) $\text{ngnscs}(X) = \min\{|X : S| : S \leq X, S \text{ normal, genuine noncongruence}\}.$

In [M4], [MS1], [MS2] we determined $\text{ncs}(\Gamma)$ in all cases, and also $\text{nncs}(\Gamma)$ [MS2, Theorem 6.2]. In this section we evaluate $\text{ngnscs}(X)$ and $\text{gncs}(X)$ in all but finitely many cases. It is clear that $\text{ngnscs}(X) \geq \text{nncs}(X)$ and that $\text{gncs}(X) \geq \text{ncs}(X)$.

An immediate consequence of Corollary 3.7 is the following.

Theorem 5.1. *Suppose that $\text{r}(X) > 0$. Then*

$$\text{ngnscs}(X) = \text{gncs}(X) = \text{nncs}(X) = \text{ncs}(X) = \text{m}(X) = 2.$$

Theorem 5.1 also holds for some but not all rank zero cases, which are listed in Theorem 3.5. For the remainder of this section we consider the cases $(g, \delta) = (1, 1), (0, 3)$ in detail. We recall from [MS4] the possible structures of the stabilizers in *any* G of the vertices of \mathcal{T} . For any $v \in \text{vert}(\mathcal{T})$ it is known that one of the following holds

- (a) $G_v \cong GL_2(k)$ or $\mathbb{F}_{q^2}^*$.
- (b) $G_v \cong k^* \times N$,
- (c) $G_v/N \cong k^* \times k^*$,

where $N \cong V^+$, the additive group of a finite dimensional k -space V . See [MS4, Corollaries 2.2, 2.4, 2.7]. We require the following.

Lemma 5.2. *Suppose that $(g, \delta) = (1, 1)$ or $(0, 3)$. Then there exist subgroups P, Q of G such that*

$$G = P *_Z Q,$$

where

- (i) $GL_2(k) \leq P$,
- (ii) $\det(Q) = k^*$.

Proof. We recall from Theorem 3.5 that in both cases $G \setminus \mathcal{T}$ is a *tree*. Let \mathcal{T}_0 be a *lift* of $G \setminus \mathcal{T}$ with respect to the natural projection of \mathcal{T} onto $G \setminus \mathcal{T}$. Hence, by definition, \mathcal{T}_0 is a subtree of \mathcal{T} isomorphic to $G \setminus \mathcal{T}$. It is known that there exists $e \in \text{edge}(\mathcal{T}_0)$ for which

$$G_e = Z.$$

The edge e naturally partitions the vertices of \mathcal{T}_0 into V_1, V_2 , say. Let $P = \langle G_v : v \in \text{vert}(V_1) \rangle$ and $Q = \langle G_v : v \in \text{vert}(V_2) \rangle$. Then from standard Bass-Serre theory [Se, p.42]

$$G = P *_Z Q.$$

We can choose \mathcal{T}_0 so that there exists $v \in \text{vert}(V_1)$ for which

$$G_v = GL_2(k).$$

In addition there exists $v \in \text{vert}(V_2)$ for which G_v is of type (a) or (c). From the descriptions of the matrices in the stabilizers of these types given in [MS4, Theorems 2.1, 2.6] it is clear that in either case

$$\det(G_v) = k^*.$$

(For stabilizers of type (a) we require the fact that the *norm* map $N_{L/k}$, where $L = \mathbb{F}_{q^2}$, is *surjective*.)

For the elliptic case $(g, \delta) = (1, 1)$ we can choose \mathcal{T}_0 so that the assertions hold. See Takahashi's paper [T], in particular [T, Theorems 3, 5].

For the case $(g, \delta) = (0, 3)$ a detailed description of \mathcal{T}_0 is given in [M5, Theorem 2.26] (for the case $d = 3$). The edge with trivial stabilizer is, in the notation of [M5], the one joining $\bar{\Lambda}_0$ and $\bar{\Lambda}_1[1]$. It turns out we can choose \mathcal{T}_0 so that the stabilizer of $\bar{\Lambda}_0$ is $GL_2(k)$ and that *all* other stabilizers of the vertices of \mathcal{T}_0 are of type (c). \square

(a) The elliptic case $(g, \delta) = (1, 1)$.

(i) Suppose that $4|q$. Then $r(\Gamma) = r(G) = 0$. By [MS2, Theorem 5.5] and Proposition 4.3

$$\text{ngnscs}(\Gamma) = m(\Gamma) = p',$$

where, with two exceptions, $p' = 3$. For the exceptional cases (when $q = 4$) $p' = 5$.

Take $M \trianglelefteq \Gamma$ with $|\Gamma : M| = p'$ and let $N = Z.M$. Then, since $Z.\Gamma = G$ it follows that $N \trianglelefteq G$ and that $|G : N| = p'$. By Proposition 4.1 and Remark 4.2 N is a normal genuine non-congruence subgroup of G . So $\text{ngnscs}(G) \leq p'$.

Now let H be any subgroup of G with $|G : H| < p'$. Then $|\Gamma : H \cap \Gamma| < p'$, and hence $\Gamma \leq H$. We conclude then that, when $4|q$,

$$\text{ngnscs}(G) = p'.$$

With the two above exceptions $\text{ngnscs}(G) = m(G) = 3$. For the two exceptional cases (when $q = 4$) we have $\text{ngnscs}(G) = \text{gncs}(G) = 5$ and $m(G) = 3$. The subgroup attaining the latter bound is (the *congruence* subgroup) Γ .

(ii) Suppose now that q is odd. From the rank formulae ([G1] or see [MS1, p.33]) it follows that here $r(\Gamma) = q$ and so Theorem 5.1 applies here for the case $X = \Gamma$. However $r(G) = 0$. With the notation of Lemma 5.2 we define an epimorphism ϕ from G to $\{\pm 1\}$ by

$$\phi(g) = \begin{cases} 1 & , \quad g \in P \\ (\det(g))^{\frac{q-1}{2}} & , \quad g \in Q \end{cases}$$

Hence there exists $N \trianglelefteq G$, containing $GL_2(k)$ for which $|G : N| = 2$. By Proposition 4.1 therefore in this case

$$\text{ngnCS}(G) = \text{m}(G) = 2.$$

We summarize the results for this case.

Theorem 5.3. *Suppose that $(g, \delta) = (1, 1)$ and that $q \neq 2$.*

(i) *If q is odd*

$$\text{ngnCS}(X) = \text{gnCS}(X) = \text{m}(X) = 2.$$

(ii) *If $4|q$ then with two exceptions*

$$\text{ngnCS}(X) = \text{gnCS}(X) = \text{m}(X) = 3.$$

For each of the exceptional cases $q = 4$ and

$$\text{ngnCS}(X) = \text{gnCS}(X) = \text{m}(\Gamma) = 5, \text{m}(G) = 3.$$

Remark 5.4. Less precise results appear to hold for the elliptic case when $q = 2$. When $q = 2$ it is known [MS1, Lemma 3.1(i)] that, for *any* A ,

$$\text{nCS}(A) = 2.$$

If A is of elliptic type and has not more than two prime ideals of degree 1 (in other words, if the underlying elliptic curve has at most 3 \mathbb{F}_2 -rational points), then by Takahashi's results [T, Theorem 5] $GL_2(A)$ has a free factor $\mathbb{Z}/3\mathbb{Z}$. Consequently, $GL_2(A)$ has a normal subgroup of index 3, which by Proposition 4.3 must be genuine. So in this case

$$2 \leq \text{gnCS}(G) = \text{ngnCS}(G) \leq 3.$$

We decide the most interesting case [Se, 2.4.4, p.115].

Example 5.5. Let

$$A = \mathbb{F}_2[x, y] \quad \text{with} \quad y^2 + y = x^3 + x + 1.$$

Then A has no prime ideals of degree 1. So by Lemma 2.5 every subgroup of index 2 is a genuine non-congruence subgroup.

(b) The case $(g, \delta) = (0, 3)$, $q \neq 2$.

(i) Suppose that $4|q$. It follows from [MS2, Theorem 4.7, Theorem 6.2] and Proposition 4.3 that

$$\text{ngnCS}(\Gamma) = \text{m}(\Gamma) = p'$$

where p' is the smallest prime dividing $q - 1$. Taking $M \trianglelefteq \Gamma$ with $|\Gamma : M| = p'$ and considering the subgroup $Z.M$ it can be shown as in (a)(i) above that

$$\text{ngnCS}(G) = \text{m}(G) = p'.$$

(ii) Suppose that q is odd. Then as in the elliptic case from Lemma 5.2 it follows that

$$\text{ngnCS}(G) = \text{m}(G) = 2.$$

We summarize the results for this case.

Theorem 5.6. *Suppose that $(g, \delta) = (0, 3)$ and that $q \neq 2$. Then*

$$\text{ngnCS}(X) = \text{gnCS}(X) = \text{m}(X) = p',$$

where p' is the smallest prime dividing $q - 1$.

6. The case $A = \mathbb{F}_q[t]$

Finally we look at the most important case $(g, \delta) = (0, 1)$, that is, $A = \mathbb{F}_q[t]$. For most aspects of Drinfeld modular curves this is the case that is by far the best understood. Ironically, for this case we don't know $\text{ngnCS}(G)$ exactly and can only give lower bounds.

Theorem 6.1. *Let $A = \mathbb{F}_q[t]$ with $q > 3$ and let N be a normal genuine non-congruence subgroup of Γ . Then $|\Gamma : N|$ is divisible by $q|PSL_2(\mathbb{F}_q)|$. In particular*

$$\text{ngnCS}(\Gamma) > \text{nncs}(\Gamma) = |PSL_2(\mathbb{F}_q)|.$$

Proof. Actually, the index of any proper normal subgroup of Γ is divisible by $|PSL_2(\mathbb{F}_q)|$. Namely, by [MS3, Lemma 3.2] we either have $\text{ql}(N) = A$ (and hence $N = \Gamma$) or $N \cap SL_2(\mathbb{F}_q) \leq \{\pm I_2\}$. In the latter case $SL_2(\mathbb{F}_q)/(N \cap SL_2(\mathbb{F}_q))$ is a subgroup of Γ/N .

So we only have to show that q^2 divides $|\Gamma : N|$. Assume not. Since $A/\text{ql}(N)$ is a subgroup of Γ/N , this implies that $\text{ql}(N)$ has codimension 1 (or 0) in A . From the previous paragraph we already know that $T(1) \notin N$, that is, $1 \notin \text{ql}(N)$. Hence there

exists a non-standard automorphism Φ of Γ with $\text{ql}(\Phi(N)) = tA$. But $\Delta(t) = \Gamma(t)$ ([M1, Corollary 1.4]); so $\Phi(N)$ is a congruence subgroup. \square

The corresponding result for $q \leq 3$ requires a little bit of preparation.

Lemma 6.2. *Let $q \leq 3$ and $A = \mathbb{F}_q[t]$. Denote the commutator group of Γ by Γ' .*

- (i) *Γ' is the normal subgroup of Γ generated by the unique subgroup of order $q^2 - 1$ of $SL_2(\mathbb{F}_q)$.*
- (ii) *If N is a normal subgroup of Γ containing Γ' , then Γ/N is naturally isomorphic to $A/\text{ql}(N)$.*
- (iii) *If N is a normal subgroup of Γ with $1 \in \text{ql}(N)$, then N contains Γ' .*

Proof. For $q \leq 3$ the commutator of $SL_2(\mathbb{F}_q)$ is its unique subgroup P of order $q^2 - 1$. So Γ' contains the normal hull of P . For the converse we use Nagao's Theorem

$$\Gamma = SL_2(\mathbb{F}_q) \underset{SB(\mathbb{F}_q)}{*} SB(A),$$

where $SB(R) = B_2(R) \cap \Gamma$. See for example [Se, p.88]. From this we see that the quotient of Γ by the normal hull of P is $T(A)$. This proves (a) and (b).

Part (c) follows from the simple fact that the normal subgroup of $SL_2(\mathbb{F}_q)$ generated by $T(1)$ is $SL_2(\mathbb{F}_q)$ itself. \square

Theorem 6.3. *Let $A = \mathbb{F}_q[t]$ with $q \leq 3$ and let N be a normal genuine non-congruence subgroup of Γ . Then q^2 divides $|\Gamma : N|$. In particular*

$$\text{ngncs}(\Gamma) > \text{nncs}(\Gamma) = q.$$

Proof. If $1 \notin \text{ql}(N)$, the proof is exactly the same as for Theorem 6.1.

If $1 \in \text{ql}(N)$ and $q^2 \nmid |\Gamma : N|$, then by Lemma 6.2 necessarily $\Gamma' \leq N$ and $|\Gamma : N| = |A : \text{ql}(N)| = q$. Then there exists an \mathbb{F}_q -vector space automorphism ϕ of A (with corresponding non-standard automorphism Φ) such that $\phi(1) = 1$ and $\phi(\text{ql}(N)) = t(t-1)A \oplus \mathbb{F}_q$. So $\Phi(N)$ has level $t(t-1)A$ (and still contains Γ').

To finish the proof we verify that $\Phi(N)$ is a congruence subgroup by showing

$$\Delta(t(t-1)).\Gamma' = \Gamma(t(t-1)).\Gamma'.$$

It is well-known that

$$\Gamma/\Gamma(t(t-1)) \cong SL_2(A/(t)) \times SL_2(A/(t-1)).$$

Under this isomorphism

$$(\Gamma(t(t-1)).\Gamma')/\Gamma(t(t-1)) \cong SL_2(\mathbb{F}_q)' \times SL_2(\mathbb{F}_q)'.$$

It follows that

$$|\Gamma : \Gamma(t(t-1)).\Gamma'| = q^2 = |\Gamma : \Delta(t(t-1)).\Gamma'|,$$

which proves the claim. \square

Theorem 6.4. *Let N be a normal genuine non-congruence subgroup of $G = GL_2(\mathbb{F}_q[t])$. Then $|G : N|$ is divisible by*

$$\begin{cases} q|PGL_2(\mathbb{F}_q)| = q^4 - q^2, & \text{if } q > 3, \\ q^2, & \text{if } q \leq 3. \end{cases}$$

In particular,

$$\text{ngncs}(G) > \text{nncs}(G).$$

Proof. First of all, $|G : N|$ is divisible by $|\Gamma : N \cap \Gamma|$, which is bigger than 1 unless $\Gamma \subseteq N$. If $q > 3$, then, as explained earlier, $|\Gamma : N \cap \Gamma|$ is divisible by $|PSL_2(\mathbb{F}_q)|$. So $|G : N| \geq \frac{q^3 - q}{2}$, and hence $|G : ZN| \geq \frac{q(q+1)}{2} > q - 1$. In particular, ZN does not contain Γ .

Now $GL_2(\mathbb{F}_q) \cap ZN$ is a normal subgroup of $GL_2(\mathbb{F}_q)$. But from the proof of Theorem 6.1 we know that $SL_2(\mathbb{F}_q) \cap ZN = \{\pm I_2\}$. This leaves only the possibility $GL_2(\mathbb{F}_q) \cap ZN = Z$. So

$$GL_2(\mathbb{F}_q)/(GL_2(\mathbb{F}_q) \cap ZN) \cong PGL_2(\mathbb{F}_q)$$

is a subgroup of $G/(ZN)$. This shows that $|G : N|$ is divisible by $q^3 - q$ if $q > 3$.

Now assume that q^2 does not divide $|\Gamma : N \cap \Gamma|$. By the proofs of Theorems 6.1 and 6.3 then there exists a non-standard automorphism Φ of Γ that maps N to a congruence subgroup. By defining Φ to act as identity on diagonal matrices, Φ extends to a non-standard automorphism Φ of G . As $\Phi(N)$ contains a congruence subgroup, N is not genuine.

To prove the last claim we exhibit a normal congruence subgroup of small index in G and quasi-level tA . By a suitable non-standard automorphism this group can then be mapped to a normal (non-genuine) non-congruence subgroup of the same index.

If $q > 3$ we can take $Z.\Gamma(t)$, which has index $|PGL_2(\mathbb{F}_q)|$. If $q = 2$, then $G = \Gamma$ anyway, and Theorem 6.3 applies. Finally, if $q = 3$ we observe that the 2-Sylow subgroup of $SL_2(\mathbb{F}_3)$ is normal in $GL_2(\mathbb{F}_3)$. Taking its inverse image under the (in this case surjective) natural map $G \rightarrow GL_2(A/(t))$, we obtain a normal subgroup of index 6. \square

Remark 6.5. More precisely, Theorems 6.1, 6.3 and 6.4 show that in order for a normal subgroup N to have a chance of being genuine $\text{ql}(N)$ must have at least

codimension 2 in $\mathbb{F}_q[t]$, and hence X/N must contain a subgroup isomorphic to $\mathbb{F}_q \oplus \mathbb{F}_q$.

We finish with a partial result on not necessarily normal genuine non-congruence subgroups.

Corollary 6.6. *Let $q = p$ a prime, i.e. $A = \mathbb{F}_p[t]$. Then $\text{gncs}(X) \geq 2p$, and hence in particular $\text{gncs}(\Gamma) > \text{ncs}(\Gamma)$.*

Proof. Let N be the core of H in X . If $|X : H| < 2p$, then $|X : N|$ divides $(2p - 1)!$ and is therefore not divisible by p^2 . So N cannot be genuine, and consequently neither can be H . \square

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References

- [B] H. Bass: *Algebraic K-Theory*, (Benjamin, New York, 1968).
- [D] J. L. Dyer: Automorphism sequences of integer unimodular groups, *Illinois J. Math.* **22** (1978), 1-30.
- [G1] E.-U. Gekeler: Le genre des courbes modulaires de Drinfeld, *C.R. Acad. Sci. Paris* **300** (1985), 647-650.
- [G2] E.-U. Gekeler: *Drinfeld Modular Curves*, (Springer LNM 1231, Berlin Heidelberg New York, 1986).
- [HR] L. K. Hua and I. Reiner: Automorphisms of the unimodular group, *Trans. Amer. Math. Soc.* **71** (1951), 331-348.
- [JT] G. A. Jones and J. S. Thornton: Automorphisms and congruence subgroups of the extended modular group, *J. London Math. Soc. (2)* **34** (1986), 26-40.
- [L] A. Lubotzky: Free quotients and the congruence kernel of SL_2 , *J. Algebra* **77** (1982), 411-418.
- [M1] A. W. Mason: Anomalous normal subgroups of $SL_2(K[x])$, *Quart. J. Math. Oxford (2)* **36** (1985), 345-358.
- [M2] A. W. Mason: The order and level of a subgroup of GL_2 over a Dedekind ring of arithmetic type, *Proc. Roy. Soc. Edinburgh Sect. A* **119** (1991), 191-212.

- [M3] A. W. Mason: Quotients of the congruence kernels of SL_2 over arithmetic Dedekind domains, *Israel J. Math.* **91** (1995), 77-91.
- [M4] A. W. Mason: On non-congruence subgroups of the analogue of the modular group in characteristic p : Rankin memorial issues. *Ramanujan J.* **7** (2003), 141-144.
- [M5] A. W. Mason: The generalization of Nagao's theorem to other subrings of the rational function field, *Comm. Algebra* **31** (2003), 5199-5242.
- [MPSZ] A. W. Mason, A. Premet, B. Sury, P. A. Zalesskii: The congruence kernel of an arithmetic lattice in a rank one algebraic group over a local field, *J. Reine Angew. Math.* **623** (2008), 43-72.
- [MS1] A. W. Mason and A. Schweizer: The minimum index of a non-congruence subgroup of SL_2 over an arithmetic domain, *Israel J. Math.* **133** (2003), 29-44.
- [MS2] A. W. Mason and A. Schweizer: The minimum index of a non-congruence subgroup of SL_2 over an arithmetic domain. II: The rank zero cases, *J. London Math. Soc.* **71** (2005), 53-68.
- [MS3] A. W. Mason and A. Schweizer: Non-standard automorphisms and non-congruence subgroups of SL_2 over Dedekind domains contained in function fields, *J. Pure Appl. Algebra* **205** (2006), 189-209.
- [MS4] A. W. Mason and A. Schweizer: The stabilizers in a Drinfeld modular group of the vertices of its Bruhat-Tits tree: an elementary approach, *Int. J. Algebr. Comput.* **23** (2013), 1653-1683.
- [R] I. Reiner: A new type of automorphism of the general linear group over a ring, *Ann. of Math.* **66** (1957), 461-466.
- [Se] J.-P. Serre, *Trees*, (Springer, Berlin, Heidelberg, New York, 1980).
- [SV] J. Smillie and K. Vogtmann: Automorphisms of SL_2 over imaginary quadratic integers, *Proc. Amer. Math. Soc.* (2) **112** (1991), 691-699.
- [St] H. Stichtenoth: *Algebraic Function Fields and Codes (Second Edition)*, (Springer GTM 254, Berlin Heidelberg, 2009).
- [T] S. Takahashi: The fundamental domain of the tree of $GL(2)$ over the function field of an elliptic curve, *Duke Math. J.* **72** (1993), 85-97.